Introduction to Koszul duality (through examples)

by Dan

This is the first talk in a series & will be oriented toward the classical ideas.

Outline:

- general theory
- examples
  - $T(V)$ "baby"
  - $\mathrm{Sym}(V)$ classical motivation
  - $\mathcal{U}(\mathfrak{g})$ deformation of Sym + filtered
  - $\Pi(Q)$ quadratic but not always Koszul

  "preprojective algebra of a quiver Q

all these are connected w/ representation theory & hence geometry

Def A quadratic presentation of a graded algebra $A = \bigoplus_{n \geq 0} A_n$ is an isomorphism

$$\varphi : T_{A_0}(A_1)/(R) \xrightarrow{\cong} A$$

with $R \subset A_1 \underset{A_0}{\otimes} A_1$

Usually: $A_0 = k$, $A_1 = V$ a fin-dim k-vector space

(more generally, A_0 is semisimple)

Def A <u>quadratic algebra</u> is an graded algebra possessing a quadratic presentation.

Def Given $A = T_{A_0}(V)/(R)$, then its <u>quadratic dual</u> is

$$A^{quad} := T_{A_0}(V^*)/(R^\perp)$$

where $R^\perp = \mathrm{Ann}_{V \otimes V}(R) \subset V^* \otimes V^*$

Remark $(T(V))^{quad} \cong k \oplus V^*$ w/ square-zero multiplication

as $R = 0$, $(R) = 0$, $R^\perp = V^* \otimes V^*$

Def The <u>Koszul dual</u> is $A^! = \mathrm{Ext}_A(k, k)$

where k is an A-algebra via the augmentation

$$\varepsilon : A \xrightarrow{\text{project}} A_0 \cong k$$

Def A is <u>Koszul</u> if $(A^!)^! \cong A$.

there's a candidate isomorphism, it's not just abstract

Equivalent characterizations of Koszulity

① $\mathrm{Ext}_A^i(k, k[n]) = 0$ unless $i = n$, $\forall n > 0$

("two gradings coincide")

② $A^{quad} \cong A^!$

"Koszul complex"

③ There is a subcomplex of the bar complex

$$(A^! \otimes A, d)$$

bar differential restricted

that resolves k as an A-module

You can see $(3) \Rightarrow (2)$ by using the Koszul complex to compute

$$A^! = \mathrm{Ext}_A(k, k).$$

Consider trying to show ① $\Rightarrow$ ③ :

Let $i=1$. Then

$$\mathrm{Ext}^1_A(k, k[n]) = \mathrm{Ext}^0_A(A_{>0}, k[n])$$

$$= \mathrm{Hom}_A(A_{>0}, k[n])$$

$$= \mathrm{Hom}_k(S, k[n])$$

$$= 0 \quad \text{unless } n=1$$

here $S \subset A_{>0}$ is a vector space of generators:

$$A \cdot S = A_{>0}$$

Hence $S \subset A_1$.

Thus A is generated by A_1

Let's take $i=2$.

$$\text{Let } \ker(\mu_1) = \ker\left(A \otimes A_1 \xrightarrow{\overset{\text{multiplication}}{\mu}} A\right)$$

$$\text{Then } \mathrm{Ext}^2_A(k, k[n]) = \mathrm{Hom}_A(\ker(\mu_1), k[n])$$

$$= \mathrm{Hom}_k(V, k[n])$$

$$= 0 \quad \text{unless } n=2$$

$$\Rightarrow V \subset A_2 \Rightarrow A \text{ is quadratic}$$

A detailed explanation can be found in the paper of Beilinson-Ginzburg-Soergel.

The Koszul resolution begins

$$\cdots \to A\otimes(R\otimes V\cap V\otimes R)\xrightarrow{\mu\otimes 1\otimes 1} A\otimes R \xrightarrow{\mu\otimes 1\otimes 1} A\otimes V \xrightarrow{\mu} A$$

and $$K_n = \bigcap_{k=0}^{n-2} V^{\otimes k}\otimes R\otimes V^{\otimes n-k-2} \subset V^{\otimes n}$$

in general, with multiplication $d = \mu\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-1 \text{ terms}}$

In particular, the n^{th} graded piece of $A^!$ is

$$A^!_n = K_n \cong (A^{quad})_n$$

Now let's leave theory behind & do some examples

① $T_k(V)$ = free associative unital algebra on V over k

$$= \bigoplus_{n\geq 0} V^{\otimes n}, \quad V^{\otimes 0} = k$$

$$T(V)^{quad} = T(V^*)/(V^*\otimes V^*) = k\oplus V^*$$

The traditional resolution of k over $T(V)$ is

$$0 \to T(V)\otimes V \xrightarrow{\mu} T(V) \xrightarrow{\varepsilon} k = T^0(V) \to 0$$

$$f\otimes x \longmapsto f\otimes x$$

and this is the Koszul resolution.

Hence

$$\mathrm{Ext}_{T(V)}(k,k) = \mathrm{Hom}_{T(V)}\left(T(V)\otimes V \xrightarrow{\mu} T(V),\ k\right)$$

$$= \left(\mathrm{Hom}_{T(V)}(T(V)\otimes V, k) \xleftarrow{\mu^*} \mathrm{Hom}_{T(V)}(T(V), k)\right)$$

becomes the zero map

these equalities are OK since the differential is zero on RHS

$$= \left(\underset{\substack{\mathbin{\rotatebox{90}{$\cong$}}\\ V^*}}{\mathrm{Hom}_k(V,k)} \xleftarrow{0} \underset{\substack{\mathbin{\rotatebox{90}{$\cong$}}\\ k}}{\mathrm{Hom}_k(k,k)}\right)$$

Thus $T(V)^! = T(V)^{quad}$!

This Koszul resolution allows us to build a small resolution of $T(V)$ as a $T(V)$-bimodule: just tensor with $T(V)$

$$\left(0 \to T(V)\otimes V\otimes T(V) \to T(V)\otimes T(V)\right) \simeq T(V)$$

$\Rightarrow HH^*(T(V)) = 0$ for $* \geq 2$

Hence $T(V)$ has no nontrivial deformations as an associative algebra.

② $\mathrm{Sym}(V) = T(V) / (x\otimes y - y \otimes x)_{x,y\in V}$

Let's compute the quadratic dual first:

$$R^{\perp} = (x^* \otimes y^* + y^* \otimes x^*)_{x^*, y^* \in V^*}$$

$$\Rightarrow \mathrm{Sym}(V)^{quad} = T(V^*)/(R^{\perp})$$
$$= \Lambda(V^*)$$

Now let's provide the Koszul resolution.

$$K^{\bullet} = \left(0 \to \mathrm{Sym}(V) \otimes \Lambda^{\dim V} V \to \dots \to \mathrm{Sym}(V) \otimes V \to \mathrm{Sym}(V)\right) \simeq k$$

$$d(f \otimes y_1 \wedge \dots \wedge y_q) = \sum_{i=1}^{q} (-1)^{i-1} f y_i \otimes y_1 \wedge \dots \hat{y_i} \dots \wedge y_q$$

<u>Claim</u> This complex has cohomology $H^n = 0$, $n \neq 0$, $H^0 \cong k$.

<u>Cor</u>: $\mathrm{Sym}(V)$ is Koszul.

Proof There's a contracting homotopy

$$h(x_1 \cdots x_p \otimes Y) = \sum_{j=1}^{p} x_1 \cdots \hat{x}_j \cdots x_p \otimes x_j \wedge Y$$

so

$$d \circ h + h \circ d = (p+q)\, id - 0 . \qquad \square$$

Let's see the computation of $\mathrm{Sym}(V)^!$:

$$\mathrm{Ext}_{\mathrm{Sym}(V)}(k,k) = \mathrm{Hom}_{\mathrm{Sym}(V)}(K^\bullet, k)$$

all differentials are zero

$$= \left(\mathrm{Hom}_{\mathrm{Sym}(V)}(\mathrm{Sym}\, V, k) \longrightarrow \mathrm{Hom}_{\mathrm{Sym}(V)}(\mathrm{Sym}\, V \otimes V, k) \to \cdots \right)$$

$$\cong \left(k \xrightarrow{0} V^* \xrightarrow{0} \Lambda^2 V^* \to \cdots \right)$$

$$\cong \mathrm{Sym}\left(V^*[-1] \right)$$

Similarly to $T(V)$ case:

$$HH^*(\mathrm{Sym}\, V) = \Lambda^*_{\mathrm{Sym}(V)}\left(\mathrm{Der}(\mathrm{Sym}\, V) \right)$$

first example of the HKR theorem

$\Rightarrow$ $HH^2(\mathrm{Sym}\, V) \cong$ biderivations $=$ 1st order deformations of the algebra

③ $U\mathfrak{g}$ = enveloping algebra of a Lie algebra $\mathfrak{g}$

$$= T(\mathfrak{g}) \Big/ \left(x \otimes y - y \otimes x - [x,y] \right)_{x,y \in \mathfrak{g}}$$

$U\mathfrak{g}$ is filtered but not graded &

$Gr(U\mathfrak{g}) \cong Sym(\mathfrak{g})$ by PBW theorem

We might hope to view $U\mathfrak{g}$ as a deformation of $Sym\,\mathfrak{g}$ & then get a "Koszul dual" by deforming $(Sym\,\mathfrak{g})^! = \Lambda \mathfrak{g}^*$:

the "answer" is

$$(U\mathfrak{g})^! = CE^*(\mathfrak{g})$$

In general, there is a version of Koszul duality of the following form:

<u>Def</u> ~~Let~~ A be a filtered algebra A that is quadratic-linear & satisfies

(I) $R \cap V = \{0\}$

(II) $V^{\otimes 2} \cap (R\otimes V + V \otimes R) \subset R \cap V^{\otimes 2}$

It says we cannot generate more quadratic relations

You can check that Ug satisfies these.

In fact, these conditions ensure that the "dual" of the quadratic-linear relation produces a square-zero differential.

Let me foreshadow the next talk
There's also a relationship btw modules:

$$A\text{-mod} \xrightarrow{\mathrm{Ext}_A(-,k)} A^!\text{-mod}$$